

# The classification of traveling wave solutions and superposition of multi-solutions to Camassa-Holm equation with dispersion

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## Abstract

Under the traveling wave transformation, Camassa-Holm equation with dispersion is reduced to an integrable ODE whose general solution can be obtained using the trick of one-parameter group. Furthermore combining complete discrimination system for polynomial, the classifications of all single traveling wave solutions to the Camassa-Holm equation with dispersion is obtained. In particular, an affine subspace structure in the set of the solutions of the reduced ODE is obtained. More general, an implicit linear structure in Camassa-Holm equation with dispersion is found. According to the linear structure, we obtain the superposition of multi-solutions to Camassa-Holm equation with dispersion.

Keywords: classification of traveling wave solution, symmetry group, Camassa-Holm equation with dispersion, superposition of solutions

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## 1 Introduction

Camassa-Holm equation (CH) ([1],[2],[3],[4]) reads

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1)$$

which describes shallow water waves. CH equation has been studied from several aspects ([1-15]). Camassa and Holm found that this equation exhibits "peakons" when  $k = 0$ , that is, solitary wave solutions with discontinuous first derivative at their crest. This peakons have the following form

$$u = c \exp(-|x - ct|). \quad (2)$$

For arbitrary value of  $k$ , Liu et al ([13, 14]) show that the CH equation has peakons as

$$u = (c + k) \exp(-|x - ct|) - k. \quad (3)$$

Liu et al ([15]) also obtain a kind of generalized kink and anti-kink wave solutions. In 2001, based on the CH equation, Dullin et al ([16]) drive Camassa-Holm equation with dispersion, its form is as follows

$$u_t + 2ku_x + 3uu_x - \varepsilon(u_{xxt} + 2u_x u_{xx} + uu_{xxx}) + \gamma u_{xxx} = 0, \quad (4)$$

where  $k, \varepsilon$  and  $\gamma$  are constants. If  $k = \varepsilon = 0$  and  $\gamma = 1$ , then Camassa-Holm equation with dispersion becomes KdV equation. If  $\varepsilon = 1$  and  $\gamma = 0$ , Camassa-Holm equation with dispersion becomes CH equation. Dullin et al ([16]) obtain peakons in the case of  $k = \gamma = 0$ . Guo and Liu, Zhang, Tang and Yang ([17, 18, 19, 20, 21]) give some peakons in some special cases. These peakons are so called weak solutions or generalized solutions. Here we consider classical solution.

In the present paper, we give the complete classification of classical traveling wave solutions of Camassa-Holm equation with dispersion. Applying the invariance property of the reduced equation of Camassa-Holm equation with dispersion under a one-parameter group, we obtain the general solution of its reduced equation. Furthermore using the complete discrimination system for polynomial, we give the classifications of traveling wave solutions to Camassa-Holm equation with dispersion. This complete result is valuable to study the physical properties of Camassa-Holm equation with dispersion. In addition, we find a two dimensional affine subspace structure in the set of the solutions of the reduced ODE. More general, an implicit linear structure in Camassa-Holm equation with dispersion is found. According to the linear structure, we give the superposition of multi-solutions to Camassa-Holm equation with dispersion. This is an interesting result.

## 2 Classification of traveling wave solutions and superposition of multi-solution to Camassa-Holm equation with dispersion

Under the traveling wave transformation  $u = u(\xi)$ ,  $\xi = x - ct$ , Camassa-Holm equation with dispersion is reduced to the following ODE

$$(2k - c)u' + 3uu' - \varepsilon(uu''' + 2u'u'' - cu''') + \gamma u''' = 0, \quad (5)$$

integrating once yields the following equation

$$u'' + \frac{1}{2(u - c - \frac{\gamma}{\varepsilon})}(u')^2 - \frac{3u^2 - 2(c - 2k)u + c_0}{2\varepsilon(u - c - \frac{\gamma}{\varepsilon})} = 0, \quad (6)$$

where  $c_0$  is an arbitrary constant.

So we only need to solve the Eq.(6). We give the general solution of the Eq.(6) in the following:

**Lemma:** The general solution of the Eq.(6) is as follows:

$$\pm(\xi - \xi_0) = \int \sqrt{\frac{\varepsilon(u - c - \frac{\gamma}{\varepsilon})}{(u - c - \frac{\gamma}{\varepsilon})^3 + d_2(u - c - \frac{\gamma}{\varepsilon})^2 + d_1(u - c - \frac{\gamma}{\varepsilon}) + d_0}} du, \quad (7)$$

where  $\xi_0$  and  $d_0$  are two arbitrary constants, and  $d_2 = 2c + 2k + \frac{3\gamma}{\varepsilon}$ ,  $d_1 = 3(c + \frac{\gamma}{\varepsilon})^2 + 2(2k - c)(c + \frac{\gamma}{\varepsilon}) + c_0$ .

According to the above lemma, we give the classification of all single traveling wave solutions to Camassa-Holm equation with dispersion. We have the following result.

**Theorem 1:** All traveling wave solutions to Camassa-Holm equation with dispersion can be classified as follows:

Case 1:  $\varepsilon > 0$ .

Case 1.1.  $d_0 = 0$ . Denote  $\Delta = d_2^2 - 4d_1$ . There are two cases to be discussed:

Case 1.1.1: If  $\Delta = 0$ , then the corresponding solutions to the Eq.(5) are

$$u = c_1 \exp(\pm \frac{\xi}{\sqrt{\varepsilon}}) - k - \frac{\gamma}{2\varepsilon}, \quad (8)$$

where  $c_1$  is an arbitrary constant.

Case 1.1.2: If  $\Delta > 0$  or  $\Delta < 0$ , then the corresponding solutions of the Eq.(5) are

$$u = c_1 \exp(\frac{\xi}{\sqrt{\varepsilon}}) + c_2 \exp(-\frac{\xi}{\sqrt{\varepsilon}}) - k - \frac{\gamma}{2\varepsilon}. \quad (9)$$

where  $c_1$  and  $c_2$  are two arbitrary constants.

Case 1.2.  $d_0 \neq 0$ . Denote

$$\Delta = -27(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_2}{3})^2 - 4(d_1 - \frac{d_2^2}{3})^3, \quad (10)$$

$$D_1 = d_1 - \frac{d_2^2}{3}, \quad (11)$$

where  $\Delta$  and  $D_1$  make up a complete discrimination system for  $F(u) = (u + \frac{\omega}{k})^3 + d_2(u + \frac{\omega}{k})^2 + d_1(u + \frac{\omega}{k}) + d_0$ . There are the following four cases to be

discussed:

Case 1.2.1: If  $\Delta = 0, D_1 < 0$ , then we have  $F(u) = (u - \alpha)^2(u - \beta), \alpha \neq \beta$ . We take the change of the variable as follows:

$$u = \frac{\beta v^2 - c - \frac{\gamma}{\varepsilon}}{v^2 - 1}, \quad (12)$$

its inverse transformation is

$$\frac{u - c - \frac{\gamma}{\varepsilon}}{u - \beta} = v^2. \quad (13)$$

then we have

$$\pm \frac{1}{\sqrt{\varepsilon}}(\xi - \xi_0) = \ln \left| \frac{v+1}{v-1} \right| + \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}} \ln \left| \frac{v - \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}}}{v + \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}}} \right|; \left( \frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta} > 0 \right). \quad (14)$$

and

$$\pm \frac{1}{\sqrt{\varepsilon}}(\xi - \xi_0) = \ln \left| \frac{v+1}{v-1} \right| - 2\sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha}} \arctan \left( v \sqrt{\frac{\beta - \alpha}{\alpha - c - \frac{\gamma}{\varepsilon}}} \right), \left( \frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha} > 0 \right). \quad (15)$$

Case 1.2.2: If  $\Delta = 0, D_1 = 0$ , then we have  $F(u) = (u - \alpha)^3$ , the solution is as follows:

$$\pm \frac{1}{2\sqrt{\varepsilon}}(\xi - \xi_0) = \pm \sqrt{\frac{u - c - \frac{\gamma}{\varepsilon}}{u - \alpha}} + \frac{1}{2} \ln \left| \frac{\sqrt{\frac{u - c - \frac{\gamma}{\varepsilon}}{u - \alpha}} \mp 1}{\sqrt{\frac{u - c - \frac{\gamma}{\varepsilon}}{u - \alpha}} \pm 1} \right|. \quad (16)$$

Case 1.2.3: If  $\Delta > 0, D_1 < 0$ , then  $F(w) = (w - \alpha)(w - \beta)(w - \rho)$ , we suppose that  $\alpha > \beta > \rho$ . Take the change of variable

$$u = \frac{\alpha v^2 - c - \frac{\gamma}{\varepsilon}}{v^2 - 1}, \quad (17)$$

then the corresponding integral becomes

$$\pm \frac{\sqrt{(\alpha - \beta)(\alpha - \rho)}}{2\sqrt{\varepsilon}(\alpha - c - \frac{\gamma}{\varepsilon})}(\xi - \xi_0) = \int \left\{ 1 + \frac{1}{2} \left( \frac{1}{v-1} - \frac{1}{v+1} \right) \right\} \frac{1}{\sqrt{(v^2 + A)(v^2 + B)}} dv. \quad (18)$$

where  $A = \frac{\beta - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}, B = \frac{\rho - c - \frac{\gamma}{\varepsilon}}{\alpha - \rho}$ . It is easy to see that the integral (35) can be expressed by the first kind of elliptic integrals and the third kind of elliptic

integrals.

Case 1.2.4: If  $\Delta < 0$ , then we have  $F(u) = (u - \alpha)(u^2 + pu + q)$ ,  $p^2 - 4q < 0$ . We take the change of variable  $v = \frac{u - c - \frac{\gamma}{\varepsilon}}{u - \alpha}$ , the corresponding integral becomes

$$\pm \frac{\xi - \xi_0}{\sqrt{\varepsilon}(\alpha - c - \frac{\gamma}{\varepsilon})} = \int (1 + \frac{1}{v - 1}) \frac{1}{\sqrt{v(Av^2 + Bv + C)}} dv, \quad (19)$$

where  $A = p\alpha + q + \alpha^2$ ,  $B = -(2\alpha + p)(c + \frac{\gamma}{\varepsilon}) - p\alpha - 2q$ ,  $C = (c + \frac{\gamma}{\varepsilon})^2 + p(c + \frac{\gamma}{\varepsilon}) + q$ , moreover  $B^2 - 4AC < 0$ . It is easy to see that the corresponding integral (36) can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Case 2.  $\varepsilon < 0$ .

Case 2.1.  $d_0 = 0$ . Denote  $\Delta = d_2^2 - 4d_1$ . If  $\Delta \leq 0$ , then there exists no solutions at all. So we only consider the case of  $\Delta > 0$ . When  $-2k - \frac{\gamma}{2\varepsilon} - \frac{\sqrt{\Delta}}{2} < u < -2k - \frac{\gamma}{2\varepsilon} + \frac{\sqrt{\Delta}}{2}$ , we have

$$u = -2k - \frac{\gamma}{\varepsilon} \pm \frac{\sqrt{\Delta}}{2} \sin(\frac{1}{\sqrt{-\varepsilon}}(\xi - \xi_0)), \quad (20)$$

or rewriting it by another form:

$$u = -2k - \frac{\gamma}{\varepsilon} + c_1 \sin(\frac{1}{\sqrt{-\varepsilon}}\xi) + c_2 \cos(\frac{1}{\sqrt{-\varepsilon}}\xi), \quad (21)$$

where  $c_1$  and  $c_2$  are two arbitrary constants. These are periodic solutions.

Case 2.2.  $d_0 \neq 0$ .  $\Delta, D_1$  and  $F(u)$  are the same with the former. There are also the following four cases to be discussed:

Case 2.2.1: If  $\Delta = 0, D_1 < 0$ , then we have  $F(u) = (u - \alpha)^2(u - \beta), \alpha \neq \beta$ . We take the change of the variable as follows:

$$u = \frac{\beta v^2 + c + \frac{\gamma}{\varepsilon}}{1 + v^2}, \quad (22)$$

its inverse transformation is

$$\frac{u - c - \frac{\gamma}{\varepsilon}}{u - \beta} = -v^2. \quad (23)$$

then we have

$$\pm \frac{1}{\sqrt{-\varepsilon}}(\xi - \xi_0) = 2 \arctan v - \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha}} \ln \left| \frac{v - \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha}}}{v + \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha}}} \right|; (\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\beta - \alpha} > 0). \quad (24)$$

and

$$\pm \frac{1}{\sqrt{-\varepsilon}}(\xi - \xi_0) = 2 \arctan v - 2 \sqrt{\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}} \arctan\left(v \sqrt{\frac{\alpha - \beta}{\alpha - c - \frac{\gamma}{\varepsilon}}}\right), \left(\frac{\alpha - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta} > 0\right). \quad (25)$$

Case 2.2.2: If  $\Delta = 0$ ,  $D_1 = 0$ , then we have  $F(u) = (u - \alpha)^3$ , the solution is as follows:

$$\pm \frac{1}{2\sqrt{-\varepsilon}}(\xi - \xi_0) = 2 \sqrt{\frac{u - c - \frac{\gamma}{\varepsilon}}{\alpha - u}} - 2 \arctan\left(\sqrt{\frac{u - c - \frac{\gamma}{\varepsilon}}{\alpha - u}}\right). \quad (26)$$

Case 2.2.3: If  $\Delta > 0$ ,  $D_1 < 0$ , then  $F(w) = (w - \alpha)(w - \beta)(w - \rho)$ , we suppose that  $\alpha > \beta > \rho$ . Take the change of variable

$$u = \frac{\alpha v^2 - c - \frac{\gamma}{\varepsilon}}{v^2 - 1}, \quad (27)$$

then the corresponding integral becomes

$$\pm \frac{\sqrt{(\alpha - \beta)(\alpha - \rho)}}{2\sqrt{-\varepsilon}(\alpha - c - \frac{\gamma}{\varepsilon})}(\xi - \xi_0) = \int \left\{1 + \frac{1}{2}\left(\frac{1}{v - 1} - \frac{1}{v + 1}\right)\right\} \frac{1}{\sqrt{-(v^2 + A)(v^2 + B)}} dv. \quad (28)$$

where  $A = \frac{\beta - c - \frac{\gamma}{\varepsilon}}{\alpha - \beta}$ ,  $B = \frac{\rho - c - \frac{\gamma}{\varepsilon}}{\alpha - \rho}$ . If  $A > 0$  and  $B > 0$ , it is easy to see that no solution can be given. In other cases, it is easy to see that the integral (28) can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Case 2.2.4: If  $\Delta < 0$ , then we have  $F(u) = (u - \alpha)(u^2 + pu + q)$ ,  $p^2 - 4q < 0$ . We take the change of variable  $v = \frac{u - c - \frac{\gamma}{\varepsilon}}{\alpha - u}$ , the corresponding integral becomes

$$\pm \frac{\xi - \xi_0}{\sqrt{-\varepsilon}(\alpha - c - \frac{\gamma}{\varepsilon})} = \int \left(1 - \frac{1}{v - 1}\right) \frac{1}{\sqrt{v(Av^2 + Bv + C)}} dv, \quad (29)$$

where  $A = p\alpha + q + \alpha^2$ ,  $B = -(2\alpha + p)(c + \frac{\gamma}{\varepsilon}) + p\alpha + 2q$ ,  $C = (c + \frac{\gamma}{\varepsilon})^2 - p(c + \frac{\gamma}{\varepsilon}) + q$ , moreover  $B^2 - 4AC < 0$ . It is easy to see that the corresponding integral (29) can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

### 3 An implicit linear structure and superposition of multi-solutions to CH-r equation

According to the case 1.2 in Sec.2, we find an interesting fact, that is, affine superposition of two solutions  $u_1 = \exp(x - ct)$  and  $u_2 = \exp(-(x - ct))$  is also a solution to the Eq.(5). This is because the Eq.(5) has a special structure. In fact, the Eq.(5) includes two parts, one part is  $-cu' + c\varepsilon u'''$ , another is  $2ku' + 3uu' - \varepsilon uu''' - 2\varepsilon u'u'' + ru'''$ . If we take

$$-cu' + c\varepsilon u''' = 0, \quad (30)$$

$$2ku' + 3uu' - \varepsilon uu''' - 2\varepsilon u'u'' + ru''' = 0, \quad (31)$$

then from the first equation, we have  $u' = \varepsilon u'''$ , therefore the second equation becomes  $u + k + \frac{r}{2\varepsilon} = \varepsilon u''$ . It is easy to see that these two equations are compatible. Therefore, if we take  $v = u + k + \frac{r}{2\varepsilon}$ , these two equations become linear equations  $v'' = \frac{1}{\varepsilon}v$  essentially. In this case, the superposition property of the solutions  $v$  is found. Thus there is a two dimensional linear subspace in the set of the solutions of  $v$ -equations. When  $\varepsilon > 0$ , the corresponding bases are  $e_1 = \exp(x - ct)$  and  $e_2 = \exp(-(x - ct))$ . When  $\varepsilon < 0$ , the bases are  $e_1 = \sin(\frac{1}{\sqrt{-\varepsilon}}\xi)$ ,  $e_2 = \cos(\frac{1}{\sqrt{-\varepsilon}}\xi)$ . This means that there is a two dimensional affine subspace in the set of the solutions of  $u$ -equations.

More general, we find an implicit linear structure in Camassa-Holm equation with dispersion. In fact, we rewrite Camassa-Holm equation with dispersion by

$$u_t - \varepsilon u_{xxt} = \varepsilon(2u_x u_{xx} + uu_{xxx}) - \gamma u_{xxx} - 2ku_x - 3uu_x, \quad (32)$$

letting the left and the right sides both are zeros yield two equations

$$u_t - \varepsilon u_{xxt} = 0, \quad (33)$$

$$\varepsilon(2u_x u_{xx} + uu_{xxx}) - \gamma u_{xxx} - 2ku_x - 3uu_x = 0. \quad (34)$$

Integrating the above first equation, we have

$$u = \varepsilon u_{xx} + f(t), \quad (35)$$

thus  $u_x = \varepsilon u_{xxx}$ , if we take  $f(t) = -k - \frac{\gamma}{2\varepsilon}$ , then the above second equation is satisfied automatically. Therefore, we find a linear structure in Camassa-Holm equation with dispersion, that is,

$$u + k + \frac{\gamma}{2\varepsilon} = \varepsilon u_{xx}. \quad (36)$$

Let  $v = u + k + \frac{\gamma}{2\varepsilon}$ , then the above equation becomes

$$v = \varepsilon v_{xx}, \quad (37)$$

when  $\varepsilon > 0$ , its general solution is as follows

$$v(x, t) = \sum [g_i(t) \exp(\frac{x}{\sqrt{\varepsilon}} - s_i(t)) + h_i(t) \exp(-\frac{x}{\sqrt{\varepsilon}} - r_i(t))], \quad (38)$$

where  $g_i(t), h_i(t), s_i(t)$  and  $r_i(t)$  all are arbitrary functions, thereafter. The corresponding  $u$ -solution is

$$u(x, t) = \sum [g_i(t) \exp(\frac{x}{\sqrt{\varepsilon}} - s_i(t)) + h_i(t) \exp(-\frac{x}{\sqrt{\varepsilon}} - r_i(t))] - k - \frac{\gamma}{2\varepsilon}. \quad (39)$$

When  $\varepsilon < 0$ , the general  $v$ -solution is

$$v(x, t) = \sum [g_i(t) \sin(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)) + h_i(t) \cos(-\frac{x}{\sqrt{-\varepsilon}} - r_i(t))], \quad (40)$$

the corresponding  $u$ -solution is

$$u(x, t) = \sum [g_i(t) \sin(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)) + h_i(t) \cos(-\frac{x}{\sqrt{-\varepsilon}} - r_i(t))] - k - \frac{\gamma}{2\varepsilon}. \quad (41)$$

Through above illustration, we show easily the affine superposition of multi-solutions to Camassa-Holm equation with dispersion using implicit linear structure. In summary, we have the following conclusions.

**Theorem 2:** When  $\varepsilon > 0$ , Camassa-Holm equation with dispersion has solution as follows:

$$u(x, t) = \sum [g_i(t) \exp(\frac{x}{\sqrt{\varepsilon}} - s_i(t)) + h_i(t) \exp(-\frac{x}{\sqrt{\varepsilon}} - r_i(t))] - k - \frac{\gamma}{2\varepsilon}; \quad (42)$$

when  $\varepsilon < 0$ , the corresponding solution is

$$u(x, t) = \sum [g_i(t) \sin(\frac{x}{\sqrt{-\varepsilon}} - s_i(t)) + h_i(t) \cos(-\frac{x}{\sqrt{-\varepsilon}} - r_i(t))] - k - \frac{\gamma}{2\varepsilon}. \quad (43)$$

where  $g_i(t), h_i(t), s_i(t)$  and  $r_i(t)$  all are arbitrary functions.

## 4 Conclusions

In summary, according to the invariance property of the Eq.(5) under the one-parameter group, we reduce it to the first order ODE and give its general solutions. Furthermore applying complete discrimination system for polynomial, we obtain the classifications of all single traveling wave solutions to Camassa-Holm equation with dispersion. Those complete results can't be given by any other indirect methods. In addition, we find that a two dimensional affine subspace structure in the set of solutions of the reduced ODE can be obtained. More general, we find an implicit linear structure. Using the linear structure, we give easily the affine superposition of multi-solutions to Camassa-Holm equation with dispersion.

**Remark:** Because Camassa-Holm equation with dispersion becomes CH equation as  $\varepsilon = 1, r = 0$ , so we will give the corresponding results to CH equation in this parameters case.

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